# A quadratically enriched Bézout theorem for tropical curves

joint with Andrés Jaramillo Puentes

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## Example: Lines on a smooth cubic surface

- #complex lines = 27 (Cayley-Salmon 1849)

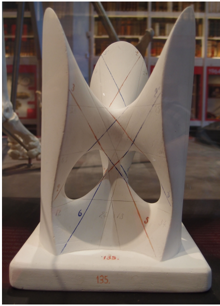


Figure: Clebsch cubic surface [1]

[1] Model in the collection of mathematical models and instruments, Georg-August-Universität Göttingen. By Oliver Zauzig. Published under CC BY-SA 3.0 on universitaetssammlungen.de

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$\mathbb{A}^1$-enumerative geometry

Bézout's theorem

Tropicalization

Tropical Bézout for curves

Enriched tropical curves

Generalizations

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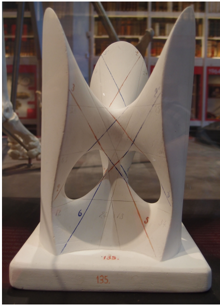
- signed count of real lines = 3 (Segre 1942)



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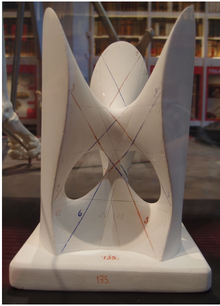

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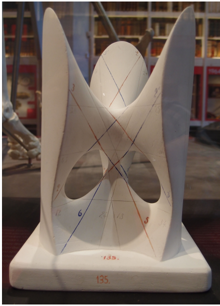
$$27 \xleftarrow{\text{rank}} \quad \xrightarrow{\text{sign}} 3$$



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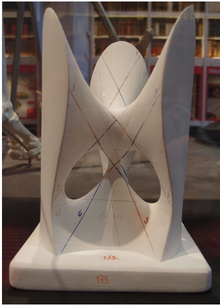
rank: $27$     sign: $3$

This is independent of the choice of smooth cubic surface.



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Let $k$ be a field of characteristic $\neq 2$.

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$\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2 \cong \{1, a\}$
$\mathrm{GW}(\mathbb{F}_p) \cong \frac{\mathbb{Z}[\langle a \rangle]}{(\langle a \rangle^2 - 1, 2\langle a \rangle - 2)}$

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**Example**

$\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2 \cong \{1, a, p, pa\}$
$GW(\mathbb{Q}_p) \cong \frac{GW(\mathbb{F}_p) \oplus GW(\mathbb{F}_p)}{(h, -h)}$

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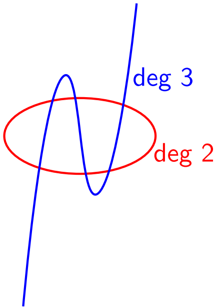
- A powerful tool to solve problems in enumerative geometry is to use tropical geometry: *tropicalalization* turns algebraic varieties into polytopes. This allows to solve problems in enumerative geometry using merely combinatorics.
- **Today**: We use tropicalization for a problem in $\mathbb{A}^1$-enumerative geometry.

$k = \mathbb{C}$
$C_1 = V(F_1) \subset \mathbb{P}^2_{\mathbb{C}}$, $d_1 = \deg F_1$
$C_2 = V(F_2) \subset \mathbb{P}^2_{\mathbb{C}}$, $d_2 = \deg F_2$

**Bézout's theorem for curves over $k = \mathbb{C}$**

$$\sum_{p \in C_1 \cap C_2} 1 = d_1 \cdot d_2$$

Today all intersections are transverse.



$k = \mathbb{R}$
$C_1 = V(F_1) \subset \mathbb{P}^2_{\mathbb{R}}$, $d_1 = \deg F_1$
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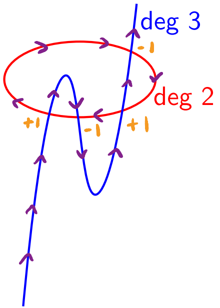
If $d_1 + d_2 \equiv 1 \mod 2$, then

$$\sum_{p \in C_1 \cap C_2} \operatorname{sign}(\det \operatorname{Jac}(F_1, F_2)(p)) = 0.$$



$k =$ arbitrary
$C_1 = V(F_1) \subset \mathbb{P}^2_k$, $d_1 = \deg F_1$
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**Bézout's theorem for curves over $k$ (McKean 2021)**

If $d_1 + d_2 \equiv 1 \mod 2$, then

$$\sum_{p \in C_1 \cap C_2} \mathrm{Tr}_{k(p)/k} \langle \det \mathrm{Jac}(F_1, F_2)(p) \rangle = \frac{d_1 \cdot d_2}{2} \cdot h \in \mathrm{GW}(k).$$

Here, $\mathrm{Tr}_{L/k}\langle a \rangle$ is the quadratic form

$$L \xrightarrow{\langle a \rangle} L \xrightarrow{\mathrm{Tr}_{L/k}} k$$

for a finite separable field extension $L/k$.

## Definition (the field of Puiseux series over $k$)

$$k\{\{t\}\} := \bigcup_{n\geq 1} k((t^{\frac{1}{n}}))$$
$$=\{a_0 t^{q_0} + a_1 t^{q_1} + \ldots | a_i \in k,$$
$$q_i \in \mathbb{Q} \text{ have a common denominator and } q_0 < q_1 < \ldots\}$$

## Lemma (Markwig-Payne-Shaw)

$\mathrm{GW}(k\{\{t\}\}) \cong \mathrm{GW}(k)$

## Proof.

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2. this defines an isomorphism $\langle a_0 t^{q_0} + \ldots \rangle \mapsto \langle a_0 \rangle$ (respects the relations in the Grothendieck-Witt rings)

□

Let

$$F(x,y) = a(t)x + b(t)y + c(t) \in k\{\{t\}\}[x,y]$$

with

$$a(t) = a_0 t^{q_{a_0}} + a_1 t^{q_{a_1}} + \dots$$
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Want to find

$x(t) = x_0 t^{-q_{x_0}} + \dots, y(t) = y_0 t^{-q_{y_0}} + \dots \in \overline{k\{\{t\}\}}$ such that

$$\begin{aligned} 0 = F(x(t), y(t)) &= a_0 x_0 t^{q_{a_0} - q_{x_0}} + \text{h.o.t.} \\ &\quad + b_0 y_0 t^{q_{b_0} - q_{y_0}} + \text{h.o.t.} \\ &\quad + c_0 t^{q_{c_0}} + \text{h.o.t.} \end{aligned}$$

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This can be solved exactly when

$$\min(q_{a_0} - q_{x_0}, q_{b_0} - q_{y_0}, q_{c_0})$$

is attained at least twice

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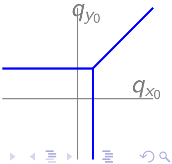
This can be solved exactly when

$$\max(-(q_{a_0} - q_{x_0}), -(q_{b_0} - q_{y_0}), -q_{c_0})$$

is attained at least twice $\rightsquigarrow$ tropical line.



$F(x,y) = \sum a_{ij}(t)x^i y^j \in k\{\{t\}\}[x,y]$ of degree $d \geq 1$ with $a_{ij}(t) = a_{ij,0}t^{q_{ij,0}} + \ldots \in k\{\{t\}\}$ has a zero $(x(t) = x_0 t^{-q_{x_0}} + \ldots, y(t) = y_0 t^{-q_{y_0}} + \ldots)$ exactly when

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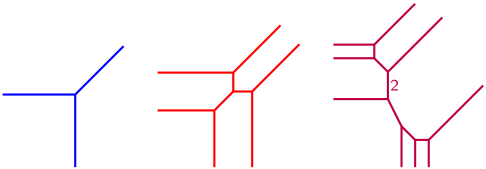


Figure: Tropical curves of degree 1, 2 and 3

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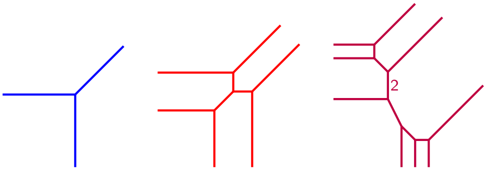


Figure: Tropical curves of degree 1, 2 and 3

Observe: degree of a tropical curve = #unbounded edges pointing to the left, down and to the upper right

$F_1, F_2 \in k\{\{t\}\}[x,y]$ of degree $d_1$ and $d_2$ $\rightsquigarrow$ tropical curves $C_1$, $C_2$, $p \in C_1 \cap C_2$

## Definition (tropical intersection multiplicity)

$\text{mult}_p(C_1, C_2) := \#$ points in $\{F_1 = F_2 = 0\}$ that "tropicalize" to $p$

## Bézout for tropical curves (Sturmfels)

Let $C_1$ and $C_2$ be two tropical curves of degree $d_1$ and $d_2$, respectively. Then

$$\sum_{p \in C_1 \cap C_2} \text{mult}_p(C_1, C_2) = d_1 \cdot d_2.$$

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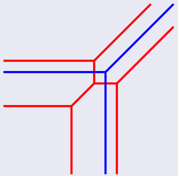

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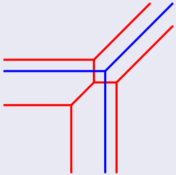

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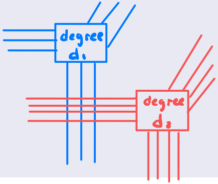

subdivision $S$ of $\Delta_d := \mathrm{Conv}\{(0,0),(d,0),(0,d)\}$

| tropical curve $C$ | dual subdivision $S$ |
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| vertices of $C$ | maximal cells in $S$ |
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such that

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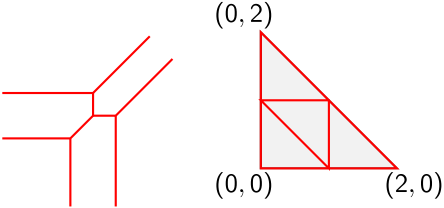


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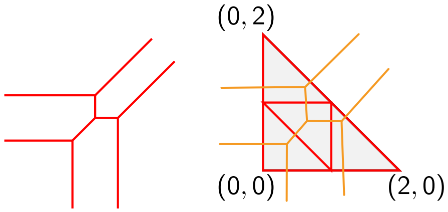


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$\text{mult}_p(C_1, C_2) := \text{Area}(\text{dual parallelogram})$

**Proof of Bézout for tropical curves.**

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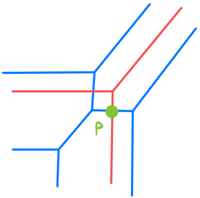


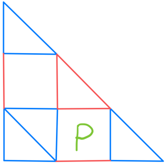

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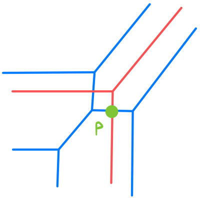


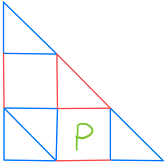

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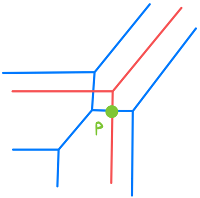


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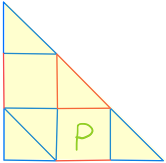

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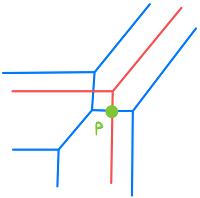


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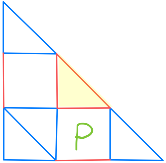

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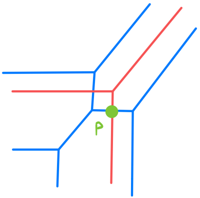


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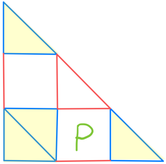

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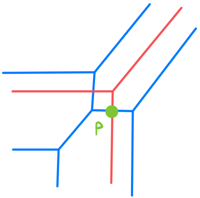


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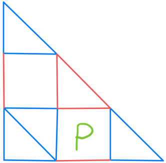

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$$\widetilde{\text{mult}}_p(C_1, C_2) := \text{Tr}_{E/k\{\{t\}\}}(\langle \det \text{Jac}(F_1, F_2)(z)\rangle) \in \text{GW}(k\{\{t\}\})$$

where $z$ is a zero of $F_1$ and $F_2$ that tropicalizes to $p$ and $E$ is the $k\{\{t\}\}$-algebra defined by all such $z$.

## Definition: Enriched tropical curve (Viro, Markwig-Payne-Shaw, Jaramillo Puentes-P.)

tropical curve with *coefficients* $a \in k^\times/(k^\times)^2$ assigned to each component/each vertex in dual subdivision

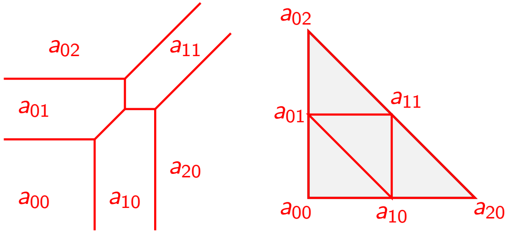


Figure: enriched tropical conic

Say $v \in \mathbb{Z}^2$ is *odd* if $v = (1,1) \in (\mathbb{Z}/2)^2$.

**Theorem (Jaramillo Puentes - P.)**

$P =$ parallelogram dual to $p \in C_1 \cap C_2$ in dual subdivision of $C_1 \cup C_2$

$$\widetilde{\text{mult}}_p(C_1, C_2) = \sum_{v \in V(P) \text{ odd}} \langle \epsilon_P(v) a_v \rangle + \frac{\text{Area}(P) - \#\{v \in V(P) \text{ odd}\}}{2} \cdot h$$

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$$\epsilon_P(v) = \begin{cases} +1 & \text{if first } C_1 \text{ then } C_2 \\ -1 & \text{if first } C_2 \text{ then } C_1 \end{cases}$$

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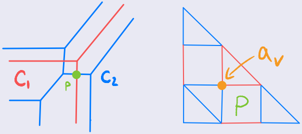


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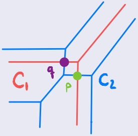


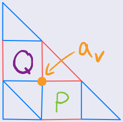


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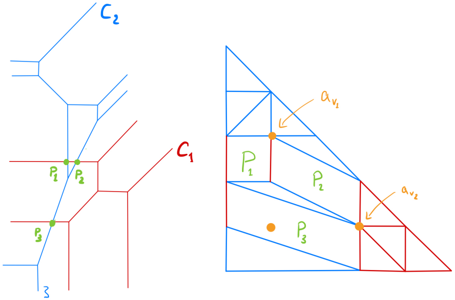
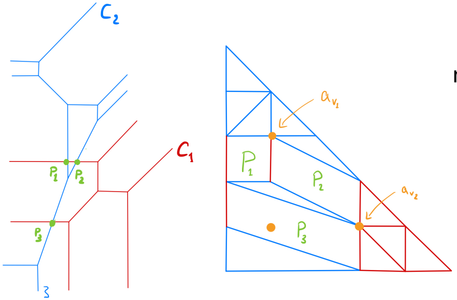



**Corollary: Quadratically enriched Bézout for tropical curves**

Assume $d_1 + d_2 \equiv 1 \mod 2$, then

$$\sum_{p \in C_1 \cap C_2} \widetilde{\text{mult}}_p(C_1, C_2) = \frac{d_1 \cdot d_2}{2} \cdot h \in \text{GW}(k)$$

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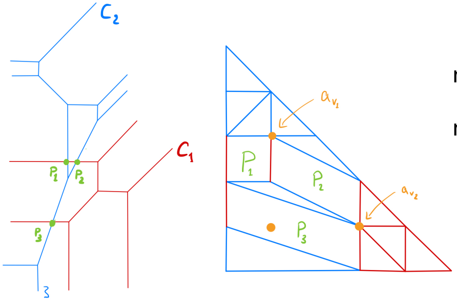


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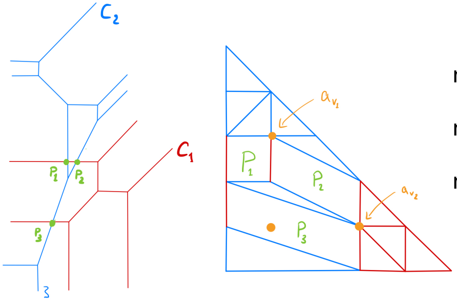


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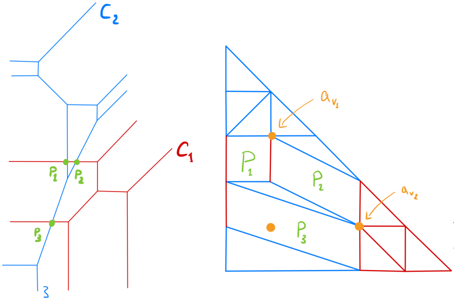


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Now the relation $\langle a_v \rangle + \langle -a_v \rangle = h$ in $\text{GW}(k)$ implies the corollary.

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