Virtual fundamental classes in motivic homotopy theory

Marc Levine: The intrinsic stable normal cone

Previously: We defined the intrinsic normal cone, the fundamental class and virtual fundamental classes in several settings.

Today: do the same in motivic homotopy theory

1) Notation
2) SH^G
3) Borel-Moore homology
4) The intrinsic stable normal cone
5) Specialization to the normal cone ⇝ fundamental class
6) p.o.t. ⇝ virtual fundamental classes
7) Example

1) Notation

B = base scheme   eg $B = \operatorname{Spec} k$

G = group scheme$/B$ $G = \{id\}$

G-resolution-property: $\forall$ finitely generated quasi-coherent G-module $\mathcal{M}$ on B $\exists$ locally free G-module $\mathcal{E} \twoheadrightarrow \mathcal{M}$

Sch_B^G = G-quasi-projective B-schemes

Sm_B^G = smooth G-quasi-projective B-schemes

$\pi_X : X \to B$ = structure map

$\mathcal{H}_\bullet^G(X)$ = equivariant unstable motivic homotopy category

$SH^G(X)$ = equivariant stable motivic homotopy category

$\mathbb{S}_B \in SH^G(B)$ sphere spectrum

2) SH^G $\quad X \in Sch_B^G$

$SH^G(X)$ is
- closed symmetric monoidal
  $1_X, \wedge_X, \mathcal{H}om_X(-,-)$
- triangulated

Six operations:

Hoyois: The six operations in equivariant motivic homotopy theory

$f: Y \to X$ in Sch_B^G

$$f^*: SH_B^G(X) \rightleftarrows SH_B^G(Y) : f_*$$

$$f_!: SH_B^G(Y) \rightleftarrows SH_B^G(X) : f^!$$

For f smooth, f^* admits a left adjoint $f_\#$

Some properties:

- f^* is symmetric monoidal
- natural transformation $f_! \to f_*$ which is an iso for f proper
- j open immersion $\Rightarrow$ $j^! = j^*$
- $$\underset{\text{closed immersion}}{Z \overset{i}{\hookrightarrow}} X \underset{\text{open complement}}{\overset{j}{\hookleftarrow} U}$$

$\rightsquigarrow$ localization distinguished triangle

$$j_! j^! \longrightarrow \mathrm{Id}_{SH^G(X)} \longrightarrow i_* i^*$$

For a G-vector bundle $p: V \to X$

$\rightsquigarrow$ $$\Sigma^V : SH^G(X) \longrightarrow SH^G(X)$$
$$\alpha \longmapsto Th_X(V) \wedge \alpha$$

$\rightsquigarrow$ Inverse $\Sigma^{-V} : SH^G(X) \to SH^G(X)$

For $f: Y \to X$ smooth

$$f_! \simeq f_\# \circ \Sigma^{-T_{Y/X}} \quad \text{and} \quad f^! \simeq \Sigma^{T_{Y/X}} \circ f^*$$

Autoequivalences:

(Ayoub, Riou for $G=\{Id\}$)
Levine for general G)

For a perfect complex $E_\bullet$

$\rightsquigarrow$ autoequivalence $\Sigma^{E_\bullet^\vee} : SH^G(X) \to SH^G(X)$

st for $f: Y \to X$

$$\Sigma^{f^* E_\bullet^\vee} \circ f^* \simeq f^* \circ \Sigma^{E_\bullet^\vee} \qquad f_* \circ \Sigma^{f^* E_\bullet^\vee} \simeq \Sigma^{E_\bullet^\vee} \circ f_*$$
$$\Sigma^{f^* E_\bullet^\vee} \circ f^! \simeq f^! \circ \Sigma^{E_\bullet^\vee} \qquad f_! \circ \Sigma^{f^* E_\bullet^\vee} \simeq \Sigma^{E_\bullet^\vee} \circ f_!$$

$E_\bullet^1 \to E_\bullet \to E_\bullet^2 \to$

$\rightsquigarrow \quad \Sigma^{E_\bullet^\vee} \cong \Sigma^{E_\bullet^{1\vee}} \circ \Sigma^{E_\bullet^{2\vee}}$

and $\quad \Sigma^{E_\bullet^\vee[1]} \cong (\Sigma^{E_\bullet^\vee})^{-1}$

Ex: $E_\bullet = (E_n \to \dots \to E_m)$

$\Rightarrow \Sigma^{E_\bullet^\vee} \cong \Sigma^{(-1)^n E_n^\vee} \circ \dots \circ \Sigma^{(-1)^m E_m^\vee}$

$\mathbb{A}^1$-homotopy invariance:

$f: V \to Z$ affine space bundle $\Rightarrow$ $f_! f^! \to \mathrm{id}$ is a natural isomorphism.

$$\begin{matrix} f_! f^! \\ \wr\| \\ f_\sharp \circ f^* \end{matrix}$$

Proper pushforward and smooth pullback

$p: Z \to W$ proper

$\pi_Z \searrow B \swarrow \pi_W$

$\rightsquigarrow$ proper pullback $\quad p^*: \pi_{W!} \to \pi_{Z!} \circ p^*$

$$\begin{tikzcd} \pi_{W!} \arrow[r] \arrow[d] & \pi_{Z!} \circ p^* \\ \pi_{W!} p_* p^* \arrow[r] & \pi_{W!} p_! p^* \arrow[u] \end{tikzcd}$$

$f: Z \to W$ smooth

$\rightsquigarrow$ Smooth pushforward $\quad f_*: \pi_{Z!} \circ \Sigma^{T_{Z/W}} \circ f^* \to \pi_{W!}$

If f is an affine space bundle

f_* is an isomorphism.

For a section s of f one can define

$s_*: \pi_{W!} \to \pi_{Z!} \circ \Sigma^{T_{Z/W}} \circ f^*$ s.t. $f_* \circ s_* = id$

3) Borel-Moore $\mathcal{E}$-homology

$\mathcal{E} \in SH^G(B)$ commutative monoid

$$\mathcal{E}^{a,b}(-) = \mathrm{Hom}_{SH(B)}\left(-, \mathcal{E} \wedge_B (S^1)^{a-b} \wedge_B \mathbb{G}_m^{b}\right)$$

Borel-Moore motive $Z/B_{B.M} := \pi_{Z!} 1_Z \in SH^G(B)$

$Z \hookrightarrow M$ $\qquad \simeq \pi_{M!}(M/M-Z)$

Borel-Moore $\mathcal{E}$-homology $\mathcal{E}^{B.M.}_{a,b}(Z) := \mathcal{E}^{-a,-b}(Z/B_{B.M.})$

Examples: $G = \{Id\}$, $B = \mathrm{Spec}\, K$

1) $\mathcal{E} = H\mathbb{Z}$ = ring spectrum representing motivic cohomology / higher Chow groups

$H\mathbb{Z}$ is oriented that means for $V \to Z$ a rk r VB / perfect complex $E_\bullet$ of virtual rk r

we get Thom isomorphisms

$$H\mathbb{Z}^{a,b}(\pi_{Z!}\Sigma^{V}\alpha) \cong H\mathbb{Z}^{a-2r,b-r}(\pi_{Z!}\alpha)$$

$\alpha \in SH(Z)$

$$H\mathbb{Z}^{a,b}(\pi_{Z!}\Sigma^{E_\bullet^\vee}\alpha) \cong H\mathbb{Z}^{a-2r,b-r}(\pi_{Z!}\alpha)$$

The virtual fundamental class will live in

$$\begin{tikzcd} H\mathbb{Z}^{0,0}(\pi_{Z!}\Sigma^{E_\bullet^\vee}1_Z) \arrow[r, phantom, "\cong"] & H\mathbb{Z}^{B.M.}_{2r,r}(Z) \arrow[d, phantom, "\|"] \\ & CH_r(Z) \end{tikzcd}$$

$$\widetilde{H\mathbb{Z}}$$

$$/\!/$$

2) $\mathcal{E} = H_0(\$_k)$

represents Milnor-Witt K-theory

i.e. for $X \in Sm_k$

$$H_0(\$_k)^{a+b,b}(X) = H^a_{Nis}(X, K^{MW}_b)$$

$$H_0(\$_k)^{a+b,b}(\pi_{X\#}\Sigma^{E_\bullet}1_X) = H^a_{Nis}(X, K^{MW}_b(\det E_\bullet))$$

The virtual fundamental class will live in

$$\begin{tikzcd} H(\$_k)^{0,0}(\pi_{Z!}\Sigma^{E_\bullet^\vee}1_Z) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\lambda\ \wr\wr"] & \widetilde{CH}_r(Z, \det E_\bullet) \arrow[d, phantom, "\wr\wr"] \\ H(\$_k)^{B.M.}_{2r,r}(Z) & \widetilde{CH}_r(Z) \end{tikzcd}$$

$H_0(\$_k)$ is SL-oriented ie an iso

$$\lambda : \det E_\bullet \to \underset{\substack{\uparrow \\ \text{line bundle} \\ \text{on } Z}}{\mathcal{L}^{\otimes 2}} \rightsquigarrow \text{Thom isomorphisms}$$

4) The intrinsic stable normal cone

Set up for the rest of the talk

$Z \in \mathrm{Sch}_B^G$

$i : Z \hookrightarrow M$ closed immersion, with ideal sheaf $\mathcal{I}_Z$, $M \in \mathrm{Sm}_B^G$

$$\mathfrak{C}_i := \mathrm{Spec} \bigoplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1}$$

$$\begin{tikzcd}
\mathfrak{C}_i \arrow[r, "p_i"] \arrow[rr, bend right, "\mathcal{G}_i"'] \arrow[dr, "\pi_{\mathfrak{C}_i}"'] & Z \arrow[r, hook, "i"] \arrow[d, "\pi_Z"] & M \arrow[dl, "\pi_M"] \\
& B &
\end{tikzcd}$$

Intrinsic stable normal cone $\mathfrak{C}_Z^{st} \in SH^G(B)$

comes with an isomorphism

$$\begin{tikzcd}
\mathfrak{C}_Z^{st} \arrow[r, "\alpha_i", "\cong"'] \arrow[dr, "\alpha_{i'}"'] & \pi_{\mathfrak{C}_i !} \circ \Sigma^{\mathcal{G}_i^* T_{M/B}} 1_{\mathfrak{C}_i} \arrow[d, "\cong", "\text{Lemma 2.1}"'] \\
& \pi_{\mathfrak{C}_{i'} !} \circ \Sigma^{\mathcal{G}_{i'}^* T_{M/B}} 1_{\mathfrak{C}_{i'}}
\end{tikzcd}$$

Ex: $Z \in Sm^q_B$, $M = Z$, $i = id$

Then $\mathbb{C}^{st}_Z \simeq \pi_{Z!} \circ \Sigma^{T_{Z/B}} 1_Z = \pi_{Z\#} 1_Z$

$$\|$$

$$Z \in SH^q(B)$$

5) Specialization to the normal cone and the fundamental class

Goal: define a fundamental class

$$[\mathbb{C}^{st}_Z]_{\mathcal{E}} \in \mathcal{E}\cancel{\mathbb{S}}^{0,0}_B(\mathbb{C}^{st}_Z)$$

So we need a $\mathbb{C}^{st}_Z \longrightarrow \mathbb{S}_B \longrightarrow \mathcal{E}$

commutative monoid

Ex: $\mathcal{E} = H\mathbb{Z}$

$$[\mathbb{C}^{st}_Z]_{H\mathbb{Z}} \in H\mathbb{Z}^{0,0}(\mathbb{C}^{st}_Z)$$

$$\|$$

$$CH_{d_{C_i}}(C_i)$$

Deformation Space

$$\begin{tikzcd}
 & \mathrm{Def}(i) := \mathrm{Bl}_{Z\times 0} M\times \mathbb{A}^1 \setminus \mathrm{Bl}_Z M \arrow[r, "p"] & M\times \mathbb{A}^1 \\
C_i \arrow[ur, hook, "i_{C_i}"] & M\times(\mathbb{A}^1-0) \arrow[u, hook, "j"] &
\end{tikzcd}$$

Fundamental Class:

localization distinguished triangle

$$\hat{i}_! \hat{j}^! \to \mathrm{Id}_{SH^{q}(\mathrm{Def}(i))} \to i_{C_i *} i_{C_i}^*$$

+ twist by $\Sigma^{p^* p_1^* T_{M/B}}$

$$\begin{tikzcd}
 & M\times \mathbb{A}^1-0 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
M & & \mathbb{A}^1-0
\end{tikzcd}$$

+ pushforward by $\pi_{\mathrm{Def}(i)!}$

⇝ distinguished triangle

$$\Sigma_T^{-1} \underset{\|}{M} \times (\mathbb{A}^1-0)_+$$

$T = S^1 \wedge \mathbb{G}_m$

$$\pi_{M\times(\mathbb{A}^1-0)!} \Sigma^{p_1^* T_{M/B}} 1_{M\times(\mathbb{A}^1-0)}$$

$$\to \pi_{\mathrm{Def}(i)!} \Sigma^{p^* p_1^* T_{M/B}} 1_{\mathrm{Def}(i)}$$

$$\to \pi_{C_i !} \circ \Sigma^{g_i^* T_{M/B}} 1_{C_i}$$

$$\begin{tikzcd}
\pi_{C_i!} \circ \Sigma^{g_i^* T_{M/B}} 1_{C_i} \arrow[r, "\partial"] & \cancel{\Sigma_{S^1}} \cancel{\Sigma_{S^1}^{-1}} \Sigma_{\mathbb{G}_m}^{-1} (M \times \mathbb{A}^1 - 0)_+ \arrow[d] \\
C_{\mathbb{Z}}^{st} \arrow[u, "\alpha_i"] & \Sigma_{\mathbb{G}_m}^{-1} \mathbb{G}_m = \mathbb{S}_B
\end{tikzcd}$$

$\rightsquigarrow \quad C_{\mathbb{Z}}^{st} \to \mathbb{S}_B \in \mathbb{S}_B^{0,0}(C_{\mathbb{Z}}^{st})$

$=:$ fundamental class $[C_{\mathbb{Z}}^{st}]$

For a commutative monoid $\mathcal{E}$ we get $e_{\mathcal{E}} \circ [C_{\mathbb{Z}}^{st}] \in \mathcal{E}^{0,0}(C_{\mathbb{Z}}^{st})$

↖unit

$=: [C_{\mathbb{Z}}^{st}]_{\mathcal{E}}$

Rmk: Lemma 3.1 shows that this is well-defined

Ex: $\mathcal{E} = H\mathbb{Z}$, $B = \operatorname{Spec} k$, $G = \{id\}$

∂ induces a map

$$H\mathbb{Z}^{1,1}(M \times \mathbb{A}^1 - 0) \to H\mathbb{Z}^{0,0}(\pi_{C_i!} \circ \Sigma^{g_i^* T_{M/B}} 1_{C_i})$$

$$\overset{\cup}{[t]} \longmapsto \partial[t] \quad ||$$

$$\overset{||}{[C_i]} \in CH_{d_{C_i}}(C_i)$$

6) Perfect obstruction theories and virtual fundamental class

$[\Phi]: E_\bullet \to \mathbb{L}_{Z/B}$ be a p.o.t. on Z of virtual rank r.

Goal: Define a virtual fundamental class

$$[Z,[\underline{\Phi}]]^{vir}_{\mathcal{E}} \in \cancel{\mathcal{B}}^{0,0}_{\cancel{B}}\left(\pi_{Z!}\circ\Sigma^{E_\bullet^\vee} 1_Z\right)$$

$$\mathcal{E}$$

$$|| \quad \mathcal{E} = H\mathbb{Z}$$

$$CH_r(Z)$$

There is $\underline{\Phi}: (F_1 \to F_0) \to (I_Z/I_Z^2 \to i^*\Omega_{M/B})$

- iso on h_0
- surj on h_1

representing $[\Phi]$

Normalized representative

$(F_\bullet, \Phi)$ is normalized if Φ_0, Φ_1 are surjective

G-resolution property => this exists

Assume $(F_\bullet, \Phi)$ is normalized

$$\Phi_1 : F_1 \twoheadrightarrow I_Z / I_Z^2$$

$$\rightsquigarrow \quad \mathrm{Sym}^* F_1 \twoheadrightarrow \bigoplus_{n \geq 0} I_Z^n / I_Z^{n+1}$$

$$\begin{tikzcd}
i_\Phi : C_i \arrow[rr, hook] \arrow[rd, "p_i"'] \arrow[rrdd, bend right, "G_i"'] & & \mathrm{Spec}\,\mathrm{Sym}^* F_1 =: F' \arrow[ld, "p_{F'}"] \\
& Z \arrow[ru, bend left, "s_0"] \arrow[rd, "i"] & \\
& & M
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_{Z!} \circ \Sigma^{-\mathcal{F}'} \circ \Sigma^{i^* T_{M/B}} 1_Z \arrow[d, "s_{0*}"] \\
\pi_{\mathcal{F}'!} \circ \Sigma^{T_{\mathcal{F}'/Z}} p_{\mathcal{F}'}^* \Sigma^{-\mathcal{F}'} \Sigma^{i^* T_{M/B}} 1_Z \arrow[d, phantom, "\cong"] \\
\pi_{\mathcal{F}'!} \circ \Sigma^{p_{\mathcal{F}'}^* i^* T_{M/B}} 1_{\mathcal{F}'} \arrow[d, "i_\Phi^*"] \\
\pi_{C_i!} \circ i_\Phi^* \circ \Sigma^{p_{\mathcal{F}'}^* i^* T_{M/B}} 1_{\mathcal{F}'} \arrow[d, phantom, "\cong"] \\
\pi_{C_i!} \circ \Sigma^{G_i^* T_{M/B}} 1_{C_i} \\
C_Z^{st} \arrow[u, "\alpha_i", "\triangleq"']
\end{tikzcd}$$

$$T_{\mathcal{F}'/Z} = p_{\mathcal{F}'}^* \mathcal{F}'$$

Ex: $\mathcal{E} = H\mathbb{Z}$, $G = \langle \mathrm{Id} \rangle$, $B = \operatorname{Spec} k$

$$s_0^* : CH_{d_{C_i}}(\mathcal{F}') \to CH_r(Z)$$

$$i_{\Phi *} : CH_{d_{C_i}}(C_i) \to CH_{d_{C_i}}(\mathcal{F}')$$

are the intersection with the zero section and proper pushforward.

Reduced normalized representatives

A normalized representative $(\mathcal{F}_\bullet, \Phi)$ of $[\Phi]$ is **reduced** if $\mathcal{F}_0 = i^* \Omega_{\mu/B}$ and $\Phi_0 = \mathrm{id}$.

Lemma 4.2/4.3 If Z is affine then $\exists$ a reduced normalized representative unqique up to a unque chain homotopy.

Reduction to the affine case

From now on assume B is affine

$$\begin{tikzcd}
\tilde{Z} \arrow[r, hook, "i_{\tilde Z}"] \arrow[d, "p_Z"'] & \tilde{M} \arrow[d, "p_M"] \\
Z \arrow[r, hook, "i_Z"'] & M
\end{tikzcd}$$

Jouanoulo device

affine space bundle

Marc defines an induced p.o.t. $(\tilde{E}_\bullet, p_M^*[\phi])$ on $\tilde{Z}$ (Lemma 4.4) and for a reduced normalized representation $(\tilde{F}_\bullet, \tilde{\Phi})$ an isomorphism (4.3)

$$\begin{tikzcd}
\pi_{Z!} \circ \Sigma^{E_\bullet^\vee} 1_Z \arrow[r, "\cong"] & \pi_{\tilde{Z}!} \circ \Sigma^{-\tilde{F}_1} \circ \Sigma^{i_{\tilde Z}^* T_{\tilde M/B}} 1_{\tilde Z} \arrow[d, "i_{\tilde\Phi}^* \circ s_{0*}"] \\
& \pi_{\tilde{\mathfrak{C}}_{i_{\tilde Z}}} \circ \Sigma^{G_{i_{\tilde Z}} T_{\tilde M/B}} 1_{\tilde{\mathfrak{C}}_{i_{\tilde Z}}} \arrow[d, phantom, "\cong"] \\
& \mathfrak{C}^{st}_{\tilde Z} \arrow[d, "\cong"] \\
& {}
\end{tikzcd}$$

Lemma 2.4

$\Phi^*_{E_\bullet,[\Phi]}$

virtual fundamental class

$$[Z,[\Phi]]^{vir} := \Phi^{\flat}_{E_\bullet,[\Phi]}([\mathbb{C}_Z^{st}]_{\mathcal{E}}) \quad y$$

$$\in \cancel{\mathcal{S}_B}^{0,0}(\pi_{Z!}\circ\Sigma^{E_\bullet^\vee}1_Z)$$

$$\begin{tikzcd} \mathbb{C}_Z^{st} \arrow[d] \\ \mathcal{S}_B \arrow[d] \\ \mathcal{E} \end{tikzcd}$$

$\mathcal{E}$

Rmk: Marc proves this is well-defined.

Comparison with Behrend-Fantechi

Z affine, $B = \operatorname{Spec} k$, $G = \{\mathrm{Id}\}$

Let $[\Phi]$ be a p.o.t. on Z with reduced normalized representative

$$\begin{tikzcd} (F_1 \arrow[r] & F_0) \arrow[r, "\Phi"] \arrow[d, phantom, "\parallel"] & (I_Z/I_Z^2 \arrow[r] & i^*\Omega_{M/B}) \\ & i^*\Omega_{M/B} & & \end{tikzcd}$$

$$\mathbb{C}_Z = [C_i/i^*T_{M/B}] \hookrightarrow N_Z = [N_i/i^*T_{M/B}] \hookrightarrow [F^1/F_0]$$

$$\uparrow$$
$$i^* T_{M/B}$$

pullback along $F' \to [F'/F^\circ]$

$$\begin{array}{ccccc} \mathfrak{C}(F^\circ) & \subset & \mathcal{N}(F^\circ) & \subset & F' \\ \parallel & & \parallel & & \\ \mathfrak{C}_i & & \mathcal{N}_i & & \end{array}$$

$$[Z,[\phi]]^{vir}_{BF} := s_0^*([\mathfrak{C}(F^\circ)])$$
$$\parallel$$
$$s_0^*([\mathfrak{C}_i]) \in CH_r(Z)$$

It follows that

$$[Z,[\phi]]^{vir}_{BF} = [Z,[\Phi]]^{vir}_{H\mathbb{Z}}$$

because 1)

$$\begin{tikzcd}
{[\mathfrak{C}_i] \in CH_{d_{\mathfrak{C}_i}}(\mathfrak{C}_i)} \arrow[d, "?"'] \\
{[C_Z^{st}]_{H\mathbb{Z}} \in H\mathbb{Z}^{0,0}(C_Z^{st})}
\end{tikzcd}$$

$$CH_{d_{\mathfrak{C}_i}}(\mathfrak{C}_i) = H\mathbb{Z}^{0,0}(\pi_{\mathfrak{C}_i!} \circ \Sigma^{\iota_i^* T_{M/B}} 1_{\mathfrak{C}_i}) \cong \alpha_i$$

2) s_{0*} and i_{Φ}^{*} induced the intersection with the zero section and proper pushforward in Chow groups, respectively.

7) Example $G = \{Id\}$, $B = \operatorname{Spec} k$

perfect

$$g = (g_1, \dots, g_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$$

st $Z = (g^{-1}(0))_{red}$ is 0-dim

$I = (g_1, \dots, g_n)$ $i : Z \hookrightarrow \mathbb{A}^n$

$$i^* \Omega^{\vee}_{\mathbb{A}^n/k} \twoheadrightarrow I/I^2$$

$$\partial/\partial x_i \mapsto g_i$$

⇝ reduced and normalized p.o.t.

$$\Phi : \left(i^* \Omega^\vee_{\mathbb{A}^n/k} \xrightarrow{d \circ g} i^* \Omega_{\mathbb{A}^n/k} \right)$$

$$\to \left(I/I^2 \xrightarrow{d} i^* \Omega_{\mathbb{A}^n/k} \right)$$

$$\begin{tikzcd}
\rightsquigarrow [Z,g]^{vir} \in \arrow[ddddd, bend right, ""] & \$_k^{0,0}\left(\pi_{Z!} \circ \Sigma^{\underbrace{i^*T_{\mathbb{A}^n/k} - i^*T^\vee_{\mathbb{A}^n/k}}_{id}} 1_Z\right) \arrow[d, equal] \\
& \$_k^{0,0}(\pi_{Z!} 1_Z) \arrow[d, equal] \\
& \$^{B.M}_{k\,0,0}(Z) \arrow[d, "\pi_{Z*}"] \\
& \$^{BM}_{k\,0,0}(\mathrm{Spec}\, k) \arrow[d, equal] \\
& \\
\deg[Z,g]^{vir} \in & GW(k)
\end{tikzcd}$$

$T_{\mathbb{A}^n/k}$ has basis $\partial/\partial x_i$

$T^\vee_{\mathbb{A}^n/k}$ has basis dx_i

Claim: $\deg[Z,g]^{vir} = \delta_{\mathbb{A}^1}(g,Z) = \sum\limits_{Z=\{x_1,\dots,x_r\}} \delta_{\mathbb{A}^1}(g,x_i)$

where $\delta_{\mathbb{A}^1}(g,Z)$ is the local $\mathbb{A}^1$-degree

$\uparrow$

see Kass-Wickelgren EKL paper

$\delta^{\mathbb{A}^1}(g,x)$ = Morel's $\mathbb{A}^1$-degree of stabilization of

small nbhd of x $\swarrow$

$$\begin{tikzcd}
S^{2n,n}_k \arrow[r, phantom, "\simeq"] & \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \arrow[r] & \mathbb{P}^n_k/(\mathbb{P}^n_k - x) \arrow[r, phantom, "\simeq"] & U/(U-x) \arrow[d, "g|_{(U,U-x)}"] \\
 & & & {}
\end{tikzcd}$$

$$S^{2n,n}_k \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \simeq \mathbb{P}^n_k/\mathbb{P}^n_k - 0$$

So $\quad S_{\mathbb{A}^1}(g,x) \in \mathrm{End}_{SH(k)}(\mathbb{1}_k) = GW(k)$

Why should this claim be true?

Let's assume g is étale in a nbhd of z.

$$\begin{tikzcd}
\mathcal{C}_i = \mathcal{C}_{Z\subset\mathbb{A}^n} \arrow[r] \arrow[d, phantom, sloped, "\cong"] & i^* T^\vee_{\mathbb{A}^n/k} = F^1 \arrow[d, phantom, sloped, "\cong"] \\
\mathbb{A}^n_Z \arrow[r, "{\substack{\uparrow \\ \text{Jacobian of } g \\ J(g)}}"'] & \mathbb{A}^n_Z
\end{tikzcd}$$

$\Rightarrow \quad \deg [Z,g]^{vir} = \mathrm{Tr}_{Z/k} \langle \det J(g) \rangle$

$= S_{\mathbb{A}^1}(g,z).$