Types of lines and dynamic Euler numbers in $GW(k)$
enriched

$\mathbb{A}^1$-Euler number

Ex $\mathbb{A}^1$-Euler characteristic from Alexey's talk

Classically: $k = \mathbb{R}$

$\pi : V \to X$ oriented rk r VB

Thom class

$$u \in H^r(V, V - s_0(X); \mathbb{Z})$$

zero section

$$\downarrow$$

$$u_F \in H^r(F, F - 0; \mathbb{Z}) = \mathbb{Z}$$

$$H^r(V, V - s_0; \mathbb{Z}) \to H^r(V; \mathbb{Z}) \xrightarrow{s_0^*} H^r(X; \mathbb{Z})$$

$$u \longmapsto e(V)$$

Euler class

If X oriented, smooth, connected closed r-mfld $\qquad r = \mathrm{rk}\, V$

and $\sigma : X \to V$ smooth general section

$$[\sigma^{-1}(0)] \in H_0(X;\mathbb{Z}) \cong \mathbb{Z}$$

$$\shortparallel$$

Poincaré dual of $e(V)$

Euler number $n(V)$

$x \in \{\sigma = 0\}$ $\quad U$ small ball around x

+ trivialization $V|_U \cong U \times \mathbb{R}^r$

$\underset{\text{basis}}{\text{dist}} \longleftrightarrow \underset{\text{basis}}{\text{standard}}$

$$\deg\left(\frac{U}{U-x} \simeq S^r \xrightarrow{\sigma} \mathbb{R}^r / \mathbb{R}^r - 0 \simeq S^r \right)$$

$$= \mathrm{ind}_x \sigma \leftarrow \text{local index}$$

Then $\displaystyle n(V) = \sum_{x \in \sigma^{-1}(0)} \mathrm{ind}_x \sigma$

Rmk: Recall from Jake's talk
$\deg f$ could also be
$f: X \to Y$ expressed as
a sum of local
contributions

$\mathbb{A}^1$-Euler number

Replace deg by $\deg^{\mathbb{A}^1}$

$\rightsquigarrow$ Euler number over arbitrary fields k

$$\deg^{\mathbb{A}^1} : \left[\mathbb{P}^r / \mathbb{P}^{r-1} , \mathbb{P}^r / \mathbb{P}^{r-1} \right]_{\mathbb{A}^1} \to GW(k)$$

Elements in here are non-degenerate symmetric bilinear form$/k$

$GW(k)$ is generated by $\langle a \rangle \simeq \left(\begin{matrix} k \times k \to k \\ (x,y) \mapsto axy \end{matrix} \right)$

$\pi : V \to X$ VB over smooth variety $/k$

A relative orientation of $\pi : V \to X$

1) $\mathcal{L} \to X$ line bundle

2) $\mathrm{Hom}(\det TX, \det V) \cong \mathcal{L}^{\otimes 2}$

ρ

$\pi : V \to X$ VB over smooth, proper variety $X/_k$ $\dim X = r = \mathrm{rk} V$

\+ rel orientation ρ

σ general section

$x \in \sigma^{-1}(0)$

Nisnevich coordinates

$\varphi : U^{\ni x} \xrightarrow{\text{étale}} \mathbb{A}^r$

st φ induces iso on $k(x)$

$V|_U \cong U \times \mathbb{A}^r$

elmt $\det V$

elmt $\det TX \longrightarrow$ elmt $\mathcal{L}^{\otimes 2}$ lol

want this to be a square

local index $\mathrm{ind}_x \tilde{\sigma} :=$

$k = k(x)$ $\qquad \deg^{\mathbb{A}^1}\left(\frac{U}{U-x} \to \mathbb{A}^r / \mathbb{A}^r - 0 \right)$

$$\frac{U}{U-x} \cong \mathbb{A}^r / \mathbb{A}^r - 0 \cong \mathbb{P}^r / \mathbb{P}^{r-1}$$

$k(x)/k$ separable

$$\mathrm{ind}_x \tilde{\sigma} = \mathrm{Tr}_{k(x)/k} \, \mathrm{ind}_{x_{k(x)}} \tilde{\sigma}_{k(x)}$$

<u>Def</u> (J-Kass, K. Wickelgren)

$\mathbb{A}^1$-Euler number

$$n(V) := \sum_{x \in \tilde{\sigma}^{-1}(0)} \mathrm{ind}_x \tilde{\sigma} \in GW(k)$$

<u>Thm</u> (T. Bachmann, K. Wickelgren)

This does not depend on $\tilde{\sigma}$.

Computation:

1) $x \in \sigma^{-1}(0)$ $\qquad \kappa(x) = k$ $\qquad$ $\kappa(x)/k$ separable

$\rightsquigarrow (f_1, \dots, f_r) : U \xrightarrow{\sigma} V|_U \cong U \times \mathbb{A}^r \to \mathbb{A}^r$

$$J(x) := \det \frac{\partial f_i}{\partial x_j}(x) \neq 0$$

coordinates

Then $\operatorname{ind}_x \sigma = \operatorname{Tr}_{\kappa(x)/k}\langle J(x) \rangle \in GW(k)$

$$\left(V \times V \to \kappa(x) \xrightarrow{\operatorname{Tr}_{\kappa(x)/k}} k \right) \in GW(k)$$

2) If $J(x) = 0$ $\rightsquigarrow$ $\exists$ KL-form

Eisenbud-Levine

Khimshiashvili

3) $\sigma^{-1}(0) \subseteq U \subseteq X$ st

$U \cong \mathbb{A}^r = \operatorname{Spec} k[x_1, \dots, x_r]$

$$\frac{k[x_1,\dots,x_r]}{(f_1,\dots,f_r)} =: E = L_1 \times \dots \times L_s$$

L_i/k separable FE

finite + étale

there are finitly many pts in $\sigma^{-1}(0)$

"no double zeros"

$$n(V) = \mathrm{Tr}_{E/k}\langle J \rangle$$

$$J =_{def} \frac{\partial f_i}{\partial x_j}$$

Computing $\mathbb{A}^1$-Euler numbers with Macaulay2

Thm (Bachmann - K. Wickelgren)

V defined over $\mathbb{Z}[\frac{1}{2}]$

$$n(V) = \frac{n_{\mathbb{C}} + n_{\mathbb{R}}}{2}\langle 1\rangle + \frac{n_{\mathbb{C}} - n_{\mathbb{R}}}{2}\langle -1\rangle$$

or

$$n(V) = \frac{n_{\mathbb{C}} + n_{\mathbb{R}}}{2}\langle 1\rangle + \frac{n_{\mathbb{C}} - n_{\mathbb{R}}}{2}\langle -1\rangle$$

$$+ \langle 2\rangle \quad - \langle 1\rangle$$

So if we know:

- $n_{\mathbb{C}}$ = rank $n(V)$
- $n_{\mathbb{R}}$ = signature $n(V)$
- disc over finite field s.t. 2 is not a square

$\Rightarrow$ Then we know everything

Assume $\partial^{-1}(0) \subseteq U \cong \mathbb{A}^r$

- rank $= \dim_k E \leftarrow \dfrac{k[x_1,\dots,x_r]}{(f_1,\dots,f_r)}$

- sgn : $k = \mathbb{R}$

$$\mathrm{sgn}\left(\mathrm{Tr}_{E/\mathbb{R}} \langle J \rangle\right)$$

- $\mathrm{disc}\left(\mathrm{Tr}_{E/\mathbb{F}_p} \langle J \rangle\right)$

$k = \mathbb{F}_p$

2 is not a square in $\mathbb{F}_p$ |

$$E = L_1 \times \ldots \times L_s$$

<u>Example</u>: Lines on a cubic surface (J Kass
K. Wickelgren)

$\mathrm{Sym}^3 S^*$ $\quad (\mathrm{Sym}^3 S^*)_{[\ell]}$ = degree 3 polynomials on ℓ

$\downarrow$

$[\ell] \in Gr(2,4)$ = 2-planes in k^4
or lines in $\mathbb{P}^3$

Let $X = \{ \underset{\swarrow \text{degree } 3}{f} = 0 \} \subseteq \mathbb{P}^3$ be cubic surface

$\rightsquigarrow$ section $\sigma_f : Gr(2,4) \to \mathrm{Sym}^3 S^*$

$[\ell] \longmapsto f|_\ell$

$$\sigma_f([\ell]) = 0 \iff f|_\ell = 0 \iff \ell \subseteq X$$

So $n(\mathrm{Sym}^3 S^*)$ counts lines on cubic surface

$$\left.\begin{array}{l} rk = 27 \\ disc = 1 \\ sgn = 3 \end{array}\right\} \begin{array}{c} 15\langle 1\rangle + 12\langle -1\rangle \\ \in GW(k) \end{array}$$

Lines on a quintic 3 fold

$$\begin{array}{c} \mathrm{Sym}^5 S^* \\ \downarrow \\ Gr(2,5) \end{array} \qquad (\mathrm{Sym}^5 S^*)_{[\ell]} = \text{degree 5 polynomials on } \ell$$

$Gr(2,5)$ = lines in $\mathbb{P}^4$

$n(\mathrm{Sym}^5 S^*)$ counts lines on a quintic 3-fold.

Problem: • too hard for my computer

• nice quintic 3-fold contain infinitly many lines

Dynamic Euler number

$$\begin{array}{ccc} n(V_k) & \longmapsto & n(V_{k((t))}) \\ \Uparrow & & \Uparrow \\ GW(k) & \longleftrightarrow & GW(k((t))) \end{array}$$

$n(V_{k((t))})$ = dynamic Euler number

Albano–Katz : $k = 8$

lines on Fermat quintic 3-fold

$$\underbrace{\{X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 = 0\}}_{= F} \subseteq \mathbb{P}^4$$

look like

$$(u : -\zeta_a u : v : -\zeta_b' v : 0)$$

$$(u : -\zeta u : av : bv : cv)\quad a\ b\ c$$

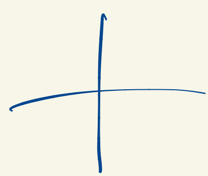
u, v coordinates on ℓ

ζ 5th root of unity

$$a^5 + b^5 + c^5 = 0$$

$$\{\sigma_F = 0\} = \bigcup_{50} C \subseteq Gr(2,5)$$

$$50 = \binom{5}{2} \cdot 5$$



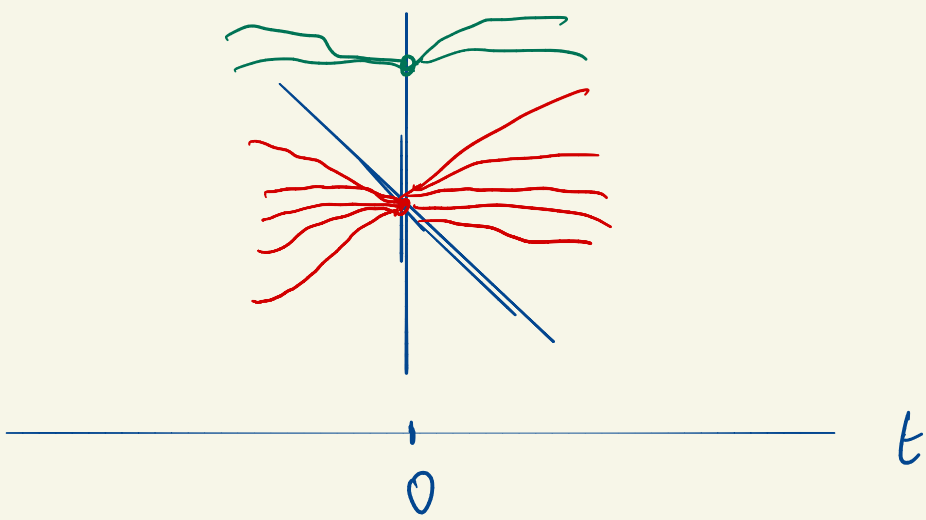
Albano-Katz study $\quad k = \mathbb{C}$

$$X_t = \{ F + tG + t^2 H + \dots = 0 \}$$

- lines in the intersection of 2 components deform with multiplicity 5 $\rightsquigarrow 2\mathbb{H} + \langle 1 \rangle$

  $\mathbb{H} = \langle 1 \rangle + \langle -1 \rangle$

- On each component there are 10 lines that deform with multiplicity 2 $\rightsquigarrow \mathbb{H}$



Thm (P.) char $k \neq 2, 5$ this still works.

$$375 \cdot 5 + 10 \cdot 2 \cdot 50 = 2875$$

$$15 \cdot (2H1 + \langle 1 \rangle) + 90 \operatorname{Tr}_{k((t))(\eta)/k((t))}(2H + \langle 1 \rangle)$$

$$+ 50 \cdot 10 \cdot H1$$

$$= 1445 \langle 1 \rangle + 1430 \langle -1 \rangle$$

$$\in GW(k((t))) \langle a \rangle$$

$$n(\mathrm{Sym}^5 S^*) \in GW(k) \langle a \rangle$$

$$\Rightarrow n(\mathrm{Sym}^5 S^*) = 1445 \langle 1 \rangle + 1430 \langle -1 \rangle$$

Rmk: This agrees M. Levine result.

Types of lines on a quintic 3-fold

char $k \neq 2$

$k = k(\ell)$

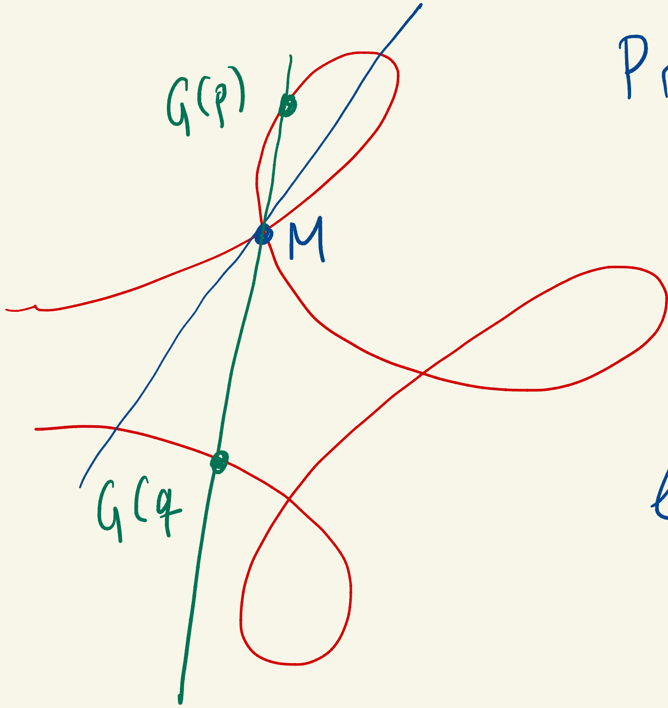
Q: What geometric information does $\mathrm{ind}_\ell \partial_f$ contain?

$\ell \subseteq X = \{f=0\} \subseteq \mathbb{P}^4$

$$\text{Gauß map} \quad G : \ell = \mathbb{P}^1 \xrightarrow{\text{degree } 4} \hat{\mathbb{P}}^2 = \text{3-planes in } \mathbb{P}^4 \text{ containing } \ell$$

$$p \longmapsto T_p X$$

Castelnuovo count G has 3 nodal pts



$P \cap X = Y$

$Y \cap Q$

$= C \cup \ell$

$\ell \cap C = \{p, q,$

$\rightsquigarrow$ Segre involution of ℓ

$$i_M : \ell \to \ell$$
$$p \mapsto q$$

fixed points of i_M are defined over $k(\sqrt{\alpha_M})$ $\alpha_M \in k^\times$

$$\mathrm{Type}(\ell) := \Big\langle \prod_{M \text{ nodal pt of } G} \alpha_M \Big\rangle \in GW(k)$$

Thm (P.) $\mathrm{Type}(\ell) = \mathrm{ind}_\ell(X)$ in $GW(k)$

$k = k(\ell)$

$$\Rightarrow \quad n(\mathrm{Sym}^3 S^*) = 1445\langle 1\rangle + 1430\langle -1\rangle = \sum_{\ell \subseteq X} \mathrm{Tr}_{k(\ell)/k} \mathrm{Type}(\ell) \in GW(k)$$

Rmk:
- Type of a line on a cubic surface is also defined via 1 segre involution (Segre/$\mathbb{R}$, Kass–Wickelgren char $k \neq 2$)

- generalizes to a definition of a type of a line on a degree $(2n-1)$ hypersurface in $\mathbb{P}^{n+1}$

(see Finashin–Kharlamov

/$\mathbb{R}$