Counting twisted cubics over an arbitrary base field k

Plan:

- Equivariant cohomology
- Atiyah-Bott localization
- Count of twisted cubics over $\mathbb{C}$ (Ellingsrud - Strømme)
- Counting in $GW(k)$
- quadratic A-B localization
- Count of twisted cubics in $GW(k)$

Equivariant Cohomology

G complex $\overset{\text{reductive}}{\text{algebraic}}$ group / compact Lie group

eg $G = T = (\mathbb{C}^*)^n$

X complex alg variety / smooth mfld

$G \circlearrowright X$

Find a contractible space EG with a free right G-action

$$X_G := \frac{EG \times X}{(e\cdot g, x) \sim (e, g\cdot x)}$$

equivariant cohomology

$$H^*_G(X) = H^*(X_G) \leftarrow \text{singular coh with } \mathbb{Z} \text{ coeff}$$

$BG = EG/G$ classifying space

Ex: $G = T \quad (\mathbb{C}^*)^m$

$ET = (\mathbb{C}^\infty \setminus \{0\})^m$

$BT = (\mathbb{CP}^\infty)^m$

$H_T^*(pt) = H^*(BT) = \mathbb{Z}[t]$

$\mathbb{Z}[t_1, \dots, t_m]$

Goal: For $V \to X$ (proper) vector bundle with $\operatorname{rank} V = \dim X$, calculate

$$H^*(pt) \ni \int_X e(V) = ev_0 \quad \int_{X_T} e_T(V) \in H^*(BT)$$

$H^*(pt) = \mathbb{Z}$, $\quad H^*(BT) = \mathbb{Z}[t_1, \dots, t_m]$

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d] & X_G \arrow[d] \\ * \arrow[r] & BG \end{tikzcd} \qquad \iota^* e_T(V) = e(V)$$

A-B: this can be expressed in terms of the fixed locus of $G \curvearrowright X$

Assume $G \curvearrowright X$ lifts to $V \to X$ (equivariant)

$$e_G(V) := e(V_G) \quad \leftarrow \frac{EG \times V}{(e \cdot g, v) \sim (e, g \cdot v)}$$

Atiyah-Bott localization $G = T = (\mathbb{C}^*)^n$

$T \curvearrowright X \qquad F =$ fixed locus

$i : F \hookrightarrow X$

<u>Location thm</u> (Atiyah-Bott)

$i_* : H_T^*(F) \to H_T^{*+2d}(X)$

$i^* : H_T^*(X) \to H_T^*(F)$

have torsion kernel and cokernel

$\mathbb{C}[t_1, \dots, t_n]$

$\uparrow \quad \shortparallel$

as $H_G^*(pt)$-modules

<u>Cor</u>: After localization they become isomorphisms

$$(i^*)^{-1} = \frac{i_*}{e_G(N_F X)}$$

$\uparrow$ normal bundle

since $i^* i_* 1 = e_G(N_F X)$

Assume $F = \{p_1, \dots, p_n\}$ $\qquad H_T^*(F) = \bigoplus_{i=1}^n H^*(BG)$

$\alpha \in$ localization of $H_G^*(X)$

$$\int_{X_G} \alpha = \int_{X_G} \frac{i_*}{e_G(N_F X)} i^* \alpha = \sum_{j=1}^{n} \frac{\cancel{i_*}}{e_G(T_{p_j} X)} \int_{p_j} i_j^* \alpha$$

$$i_j : \{p_j\} \hookrightarrow X$$

Ex: Euler characteristic

$T \curvearrowright X$ with finitely many fixed pts

$\nwarrow$ proper

Then $\chi(X) = \int_{X_T} e_T(TX) = \#$ fixed pts

eg $G = (\mathbb{C}^*)^{n+1} \curvearrowright \mathbb{CP}^n$

$$\chi(\mathbb{CP}^n) = n+1$$

fixed pts $[0 : \dots : 0 : 1 : 0 : \dots 0]$

Counting twisted cubics (Ellingsrud–Strømme)

H_n = moduli space of twisted cubic in $\mathbb{P}^n$ ← smooth rational curve of degree 3

Piene–Schlessinger: H_3 smooth + proj + dim 12

fibration: $\Phi : H_n \to G(3,n)$ ← 3-planes in $\mathbb{P}^n$

fiber is H_3

$\Rightarrow$ H_n smooth proj dim $= 12 + (3+1)\cdot(n+1-3-1) = 4n$

Notation: $x \in H_n \rightsquigarrow C_x \subset \mathbb{P}^n$ corresponding twisted cubic

$\mathcal{E}_d \to H_n$ vector bundle with fiber

$(\mathcal{E}_d)_x$ = degree d polynomials on C_x

$F \in \mathbb{C}[x_0, \ldots, x_n]_d \rightsquigarrow$ section of $\mathcal{E}_d$ defined by restricting F to C_x

zeros of the section $\longleftrightarrow$ twisted cubics on $V(F)$

If $\operatorname{rank} \mathcal{E}_d = \dim H_n$ then we get finitely many zeros of the section

$\operatorname{rank} \mathcal{E}_d = 3d+1$, $\dim H_n = 4n$

twisted cubics on $V(F)$ = # zeros of the induced section

$$= \int_{(H_n)_T} e_T(\mathcal{E}_d) = \sum_{\substack{\text{fixed} \\ \text{pts}}} \frac{1}{e_T(T_p H_n)} \int_p e_T(\mathcal{E}_d|_p)$$

$$\underset{\substack{d=5 \\ n=4}}{=} 317\,206\,375$$

here the action is induced by

$T = (\mathbb{C}^*)^{n+1} \curvearrowright \mathbb{P}^n$

Easier example: Lines on a cubic surface

$$\#\text{ lines on a cubic surface} = \int_{G(1,3)_T} e_T(\mathrm{Sym}^3 S^\vee)$$

Grassmannian of lines in $\mathbb{P}^3$

$T = (\mathbb{C}^\times)^4 \curvearrowright \mathbb{P}^3 \rightsquigarrow T \curvearrowright G(1,3)$ has $\binom{4}{2}$ fixed pts

$\ell = \{x_2 = x_3 = 0\}$ is a fixed pt

$\mathrm{Sym}^3 S^\vee|_\ell$ = VS basis $x_0^3 \oplus x_0^2 x_1 \oplus x_0 x_1^2 \oplus x_1^3$

= 4-dim T-rep

= sum of 4 irred rep

$$e_T(\mathrm{Sym}^3 S^\vee|_\ell) = e_T(\langle x_0^3\rangle)\, e_T(\langle x_0^2 x_1\rangle) \cdot e_T(\langle x_0 x_1^2\rangle)\, e_T(\langle x_1^3\rangle)$$

$$= 3t_0 \cdot (2t_0 + t_1) \cdot (t_0 + 2t_1) \cdot 3t_1$$

$T_\ell G(1,3)$: basis $x_0 \otimes e_2$, $x_1 \otimes e_2$,

$x_0 \otimes e_3$, $x_1 \otimes e_3$

$$T_\ell G(1,3) \cong \ell^\vee \otimes \mathbb{C}^4/\ell$$

$$e_T(T_\ell G(1,3)) = (t_0 - t_2)(t_1 - t_2)(t_0 - t_3)(t_1 - t_3)$$

Exercise: $$\sum_{\substack{\text{fixed} \\ \text{pts } \ell}} \frac{e_T(\mathrm{Sym}^3 S^\vee|_\ell)}{e_T(T_\ell G(1,3))} = 27$$

What about other fields k? $\operatorname{char} k \neq 2$

Motivation from motivic homotopy theory

$\Rightarrow$ We should count in $GW(k)$

Grothendieck-Witt ring of k:

= group completion of isometry classes of non-degenerate quadratic forms / k

Addition: $\oplus$ direct sum

$\otimes$ makes this into a ring

Diagonalize $\Rightarrow$ generators of $GW(k)$

$a_1x_1^2 + \dots + a_nx_n^2$ $\qquad$ $\langle a \rangle = ax^2$

$\qquad\qquad \shortparallel$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $a \in k^\times/(k^\times)^2$

$\langle a_1 \rangle + \dots + \langle a_n \rangle \in GW(k)$

relations: 1) $\langle a \rangle \cdot \langle b \rangle = \langle ab \rangle$

2) $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle$

3) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle = h$

hyperbolic form

Ex (Kass-Wickelgren):

Count of lines on a cubic surface

$$\underbrace{12h + 3\langle 1\rangle}_{} \in GW(k)$$

rank ↙ ↘ signature

27 = classical complex count

3 = signed real count

Want a "quadratic" analogue of Atiyah-Bott localization.

$$W(k) = \frac{GW(k)}{\mathbb{Z} \cdot h}$$

R ring $\rightsquigarrow$ $W(R)$ $\rightsquigarrow$ sheaf $\mathcal{W}$ on Sm_k

" smooth varieties$/k$

$X \mapsto H^*(X, \mathcal{W})$

module over $H^0(\operatorname{Spec} k, \mathcal{W}) = W(k)$

replaces $\mathbb{C}^*$

Problem: $H^*(B\mathbb{G}_m, \mathcal{W}) = W(k)$

not $W(k)[t]$

Thm (Ananyevskiy):

$$H^*(B(SL_2^n), W) = W(k)[e_1, \dots, e_n]$$

where $e_i = p_i^* e$, $e = e_{SL_2}(F)$

standard SL_2-action on 2-dim F

Thm (Levine):

N = normalizer of $\mathbb{G}_m$ in SL_2

1) N is generated by $\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

2) $H^*(BSL_2, W) \to H^*(BN, W)$ is inj.

3) $H^*(BSL_2, W)[e^{-1}] \cong H^*(BN, W)[e^{-1}]$

One more advantage:

Marc found all irred rep of N + equivariant Euler classes

these are "nice"

trivial + 2-dim

see M. Levine Motivic Euler char + Witt val charcl

Localization thm in Witt cohomology

by M. Levine

$N \curvearrowright X$ with fixed locus F ← smooth prop var $/k$

N^m for "nice" X $i : F \hookrightarrow X$

Then i_* and i^* become isos after a localization

Cor: Integration formula

Assume $N \curvearrowright X$ has finitely many fixed pts $p_1, \dots, p_n$ then

$$\int_{X_N} \alpha = \sum_{j=1}^{n} \int_{p_j} \frac{i_j^* \alpha}{e_N(T_{p_j} X)} \in W(k)$$

$\alpha \in$ localization of $H^*_N(X, \mathcal{W})$

Examples:

1) $N^m \curvearrowright \mathbb{P}^n$ $\qquad m = \left\lfloor \frac{n+1}{2} \right\rfloor \qquad N^m \hookrightarrow Gl_{n+1}$

$F = [0 : \ldots : 0 : 1]$

has $\begin{cases} 1 & n \text{ even} \\ \emptyset & n \text{ odd} \end{cases}$

$$\chi^{\mathbb{A}^1}(\mathbb{P}^n) = \begin{cases} \frac{n}{2} h + \langle 1 \rangle & n \text{ even} \\ \frac{n+1}{2} h & n \text{ odd} \end{cases} \quad \text{in } GW(k)$$

2) Lines on a cubic surface

$N^2 \curvearrowright \mathbb{P}^3$

$\rightsquigarrow N^2 \curvearrowright G(1,3)$ = Grassmannian of lines in $\mathbb{P}^3$

has 2 fixed pts $\{x_2 = x_3 = 0\} = \ell_1$

$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\{x_0 = x_1 = 0\} = \ell_2$

basis $x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3$

count of lines on a cubic surface $= \int_{G(1,3)_{N^2}} e^{N^2}(\mathrm{Sym}^3 S^\vee) = \int_{\ell_1} \frac{e_{N^2}(\mathrm{Sym}^3 S^\vee|_{\ell_1})}{e_{N^2}(T_{\ell_1} G(1,3))} + \int_{\ell_2} \frac{e_{N^2}(\mathrm{Sym}^3 S^\vee|_{\ell_1})}{e_{N^2}(T_{\ell_1} G(1,3))}$

Thm 7.1 in Levine Motivic Euler char & Witt valued char classes

$$= \frac{3e_1 \cdot e_1}{e_1^2 - e_2^2} + \frac{3e_2 \cdot e_2}{e_2^2 - e_1^2} = 3 \in W(k)$$

$\rightsquigarrow 12h + 3 \in GW(k)$

More generally: Count of lines on complete intersections $V(F_1, \dots, F_n)$ (if finitely many + orientable)

$$= d_1!! \cdot \ldots \cdot d_n!!$$

where d_i's are odd and

$$d_i!! = d_i \cdot (d_i - 2) \cdot \ldots \cdot 3 \cdot 1$$

3) Count of twisted cubics (Levine-P.)

Action above on $\mathbb{P}^n$

$\rightsquigarrow$ action on H_n

- $\int_{H_4} e(\mathcal{E}_5) = 765 \in W(k)$

  $\shortparallel$

  twisted cubics on a quintic 3-fold

  In $GW(k)$:

  $$\frac{317206375 - 765}{2} h + 765\langle 1 \rangle \in GW(k)$$

- $\int_{H_5} e(\mathcal{E}_3 \oplus \mathcal{E}_3) = 90 \in W(k)$
- $\int_{H_{10}} e(\mathcal{E}_{13}) = 768328170191602020$
- $\int_{H_{11}} e(\mathcal{E}_7 \oplus \mathcal{E}_7) = 1364980023036000$