# Arithmetic enrichments of classical results in enumerative geometry

What is arithmetic geometry?

one uses algebraic geometry to solve problems in number theory by studying algebraic varieties over arbitrary fields i.p not algebraically closed one

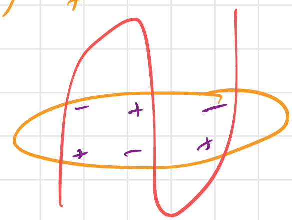
e.g. $\mathbb{Q}$, $\mathbb{F}_q$, $\mathbb{Q}_p$

over

Problems in number theory: How many solutions does a system of poly eq have

## Bézout's theorem over $\mathbb{C}$

$V_i = V(F_i) \subseteq \mathbb{CP}^n$ $\deg F_i = d_i$



Then

$$\sum_{p \in V_1 \cap \dots \cap V_n} \mathrm{mult}_p(V_1, \dots, V_n) = d_1 \cdot \ldots \cdot d_n$$

**Proof:** $V := \mathcal{O}(d_1) \oplus \dots \oplus \mathcal{O}(d_n)$

$$\begin{array}{c} V \\ \downarrow \uparrow (F_1, \dots, F_n) \\ X := \mathbb{CP}^n \end{array}$$

$\#$ intersection pts of $V_1 \cap \dots \cap V_n$ $= \#$ zeros of section $(F_1, \dots, F_n)$

$= \deg c_n(V)$

$= d_1 \cdot \dots \cdot d_n$

$\square$

**Over $\mathbb{R}$:**

$V \to X$ oriented rank $n$ vector bundle over a smooth closed oriented $n$-mfld

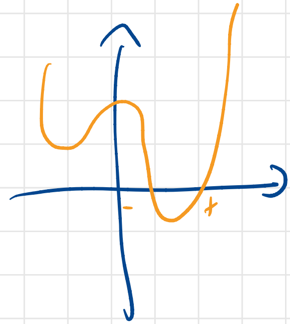
$e(V) \in H^n(X)$

fundamental class

$\deg e(V) \in \mathbb{Z} = H_n(X)$

What is $\deg e(V)$ ? $= 0$

$$\begin{array}{ccc} V & \longrightarrow & X \\ \shortparallel & & \shortparallel \\ \bigoplus \mathcal{O}(d_i) & & \mathbb{RP}^n \\ \mathcal{O}(2a) \to & , & \mathbb{RP}^1 \end{array}$$



What does this mean?

$$\begin{tikzcd} V \arrow[r] & X \arrow[l, bend left, "\sigma"] \end{tikzcd} \qquad p \in \{\sigma = 0\}$$

Choose • oriented coordinates around $p$

• trivialization of $V$ around $p$ compatible with orientation of $V$

locally around $p$,

$$\sigma : {}_{p \in} U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$\operatorname{ind}_p \sigma := \deg_p \sigma$

where $\deg_p \sigma = \deg\Big( \underbrace{U/_{U - \{p\}}}_{\simeq S^n} \xrightarrow{\sigma} \underbrace{\mathbb{R}^n/_{\mathbb{R}^n \setminus \{0\}}}_{\simeq S^n} \Big)$

$$S^n \xrightarrow{f} S^n$$

$$\rightsquigarrow \quad \begin{array}{ccc} H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} & & \mathbb{Z} \\ 1 & \longmapsto & \deg f \end{array}$$

## Poincaré-Hopf theorem

$$\deg e(V) = \sum_{p \in \{\sigma = 0\}} \operatorname{ind}_p \sigma$$

There is an analogue of the degree for an arbitrary field $k$.

Comes from "$\mathbb{A}^1$-homotopy theory"

## Morel's $\mathbb{A}^1$-degree:

$k$ arbitrary field
char $k \neq 2$

$$\deg^{\mathbb{A}^1} : [\mathbb{P}^n_k / \mathbb{P}^{n-1}_k , \mathbb{P}^n_k / \mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \to GW(k)$$

replace $\deg : [S^n, S^n] \to \mathbb{Z}$

$\mathbb{A}^1$-htpy classes

htpy classes

$f: S^n \to S^n \mapsto \deg f$

$GW(k)$: group completion of isometry classes of non-degenerate quadratic forms $/k$

$q_1 : V_1 \to k \quad , \quad q_2 : V_2 \to k$ $\Big\}$ $\Rightarrow$ monoid

$q_1 \oplus q_2 : V_1 \oplus V_2 \to k$

$\otimes \rightsquigarrow$ ring

can always diagonalize: can assume $q(x_1, \dots, x_n) = a_1 x_1^2 + \dots + a_n x_n^2$

$a_i \in k^\times = k \setminus \{0\}$

generators for $GW(k)$: $\langle a \rangle = a x^2 \quad a \in k^\times / (k^\times)^2$

relations: 1) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

2) $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle$

3) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle = h$

hyperbolic form

Examples :

- $GW(\mathbb{C}) = \mathbb{Z}$
- $GW(\mathbb{R}) = \mathbb{Z}[C_2]$

elmts of the form

$m\langle 1\rangle + n\cdot\langle -1\rangle$

$\langle 1\rangle^2 = \langle 1\rangle$

- $GW(\mathbb{F}_q) = \mathbb{Z} \oplus \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ $= \{1, s\}$

discriminant $\qquad \mathbb{Z}/2\mathbb{Z}$

- $GW(\mathbb{Q}_p) = \dfrac{\overset{\langle a\rangle}{GW(\mathbb{F}_p)} \oplus \overset{\langle ap\rangle}{GW(\mathbb{F}_p)}}{\langle h, -h\rangle}$

- $GW(\mathbb{Q})$ : complicated

$W(\mathbb{Q}) = \bigoplus_{p \text{ prime}} W(\mathbb{F}_p) \oplus W(\mathbb{R})$

$p \neq 2 \quad \oplus \mathbb{Z}/2 \qquad\qquad \text{''} \; \mathbb{Z}$

<u>Idea</u>: (Kass-Wickelgren)

Replace $\deg_p \sigma$ by $\deg_p^{\mathbb{A}^1} \sigma$ in PH theorem

<u>Def</u>: A vector bundle $V \to X$ is relatively orientable if $\exists$ line bundle $\mathcal{L} \to X$ + iso $\rho: \mathrm{Hom}(\det TX, \det V) \xrightarrow{\cong} \mathcal{L}^{\otimes 2}$

$X$: k-variety

$\mathrm{Hom}(\det TX, \det V) \cong \omega_{X/k} \otimes \det V$

- coordinates (around zero of $\sigma$) define a section of $\det TX$
- trivialization defines a section of $\det V$

$\hookrightarrow$ these are compatible with rel orientation $\rho$ if the induced section of $\mathrm{Hom}(\det TX, \det V)$ is sent to a square by $\rho$.

**Ex:** When is $V = \mathcal{O}(d_1) \oplus \dots \oplus \mathcal{O}(d_n)$
$$\downarrow$$
$$\mathbb{P}^n_k$$
relatively orientable?

$$\omega_{\mathbb{P}^n_k} \otimes \det V = \mathcal{O}(\underbrace{-n-1+d_1+\dots+d_n}_{\text{should be even}})$$

Let $V \to X$ ⟵ smooth proper $k$-variety be rel oriented vb of rank $= \dim X$ and $\sigma : X \to V$ a section with only isolated zeros

**Def** (Kass-Wickelgren)
$$\mathrm{ind}_p \sigma := \deg_p^{\mathbb{A}^1} \sigma$$ ⟵ coord + triv compatible with rel or

Euler number
$$n^{PH}(V, \rho) := \sum_{\substack{\text{zeros} \\ \text{of } \sigma}} \mathrm{ind}_p \sigma \in GW(k)$$

**Fact** (Bachmann-Wickelgren): This is

independent of the choice of the section $\sigma$.

<u>Example</u>: Lines on a smooth cubic surface $X = V(f) \subseteq \mathbb{P}^3$

deg 3

- There are always 27 lines on $X$ (independent of the choice of $X$) when counted over $\mathbb{C}$.
- Over $\mathbb{R}$: There can be 3, 7, 15 or 27 lines on $X$ (<u>depends</u> on choice of $X$) but signed count <u>always</u> equals 3
- Over $K$: Kass-Wickelgren: always get

$$\underbrace{15\langle 1\rangle + 12\langle -1\rangle}_{} \in GW(k)$$

rank $\to$ 27

signature $\to$ 3

Over $\mathbb{F}_q$: If all lines are defined over $\mathbb{F}_q$ $\Rightarrow$ # lines that contribute $\langle s \rangle$ is even

$\#\{$lines of type $\langle s \rangle$ defined over $\mathbb{F}_{q^a}$ $a$ odd$\}$

$+\ \#\{$lines of type $\langle 1 \rangle$ defined over $\mathbb{F}_{q^a}$ $a$ even$\}$

is even

Example (Bézout):

$$V = \mathcal{O}(d_1) \oplus \dots \oplus \mathcal{O}(d_n) \longrightarrow \mathbb{P}^n_k \quad (F_1, \dots, F_n)$$

is rel orientable if $\sum d_i \equiv n+1 \mod 2$

In this case

$$\sum_{p \in V(F_1) \cap \dots \cap V(F_n)} \mathrm{Tr}_{k(p)/k} \langle \det \mathrm{Jac}(F_1, \dots, F_n)(p) \rangle = \frac{d_1 \cdots d_n}{2} \cdot h \in GW(k)$$

always (McKean)

$$V \xrightarrow{a} k(p) \xrightarrow{\mathrm{Tr}_{k(p)/k}} k$$

$h = \langle 1 \rangle + \langle -1 \rangle$

What about the non-orientable case?

Not everything is possible:

<u>$n = 1$</u>: $\mathcal{O}(d) \to \mathbb{P}^1_k$ $\quad d$ odd

$$\underbrace{\langle 1 \rangle + \langle -1 \rangle + \dots \langle 1 \rangle + \langle -1 \rangle}_{d-1} + \langle a \rangle$$

$\underline{n=2}$: $\mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \to \mathbb{P}^2$

$d_1 + d_2$ even

$\langle 1 \rangle + \langle -1 \rangle + \dots + \langle 1 \rangle + \langle -1 \rangle + \langle a_1 \rangle + \dots + \langle a_d \rangle$

$d = \min(d_1, d_2)$

⋮

done using "tropical geometry"
together Jaramillo Puentes

<u>Ex</u>:

rational curves of deg $d$ on a quintic 3-fold $X = V(f) \subseteq \mathbb{P}^4$
↑ deg 5

deg 1: lines

$1445\langle 1 \rangle + 1430\langle -1 \rangle \in GW(k)$

(Levine, P.)

deg 2: cones

$\frac{609\,250}{2} \cdot h$

deg 3: twisted cubics

$$\frac{317\,206\,375 - 765}{2} \cdot h + 765 \cdot \langle 1 \rangle \in GW(k)$$

(Levine - P.)

uses "localization"